CHAPTER 7
SUCCESSIVE DIFFERENTIATION

TOPICS:

1 . Successive differentiation-nth derivative of a function – theorems.

2. Finding the nth derivative of the given function.

3. Leibnitz's theorem and its applications.

SUCCESSIVE DIFFERENTIATION

Let f be a differentiable function on an interval I. Then the derivative f′ is a function of x and if f′ is differentiable at x, then the derivative of f′ at x is called second derivative of f at x. It is denoted by f″(x) or $f^{(2)}(x)$.similarly, if f" is differentialble at x , then this derivative is called the 3rd derivative of f and it is denoted by $f^{(3)}(x)$. Proceeding in this way the nth derivative of f is the derivative of the function $f^{(n-1)}(x)$ and it the denoted by $f^{(n)}(x)$.

If y = f(x) then $f^{(n)}(x)$ is denoted by $\frac{d^n y}{dx^n}$ or $D^n y$ or $y^{(n)}$ or y_n

and $f^{(n)}(x) = \lim_{h\to 0} \frac{f^{(n-1)}(x+h) - f^{(n-1)}(x)}{h}$

THEOREM

If $f(x) = (ax + b)^m$, $m \in R$, $ax + b > 0$ and $n \in N$ then

$f^{(n)}(x) = m(m-1)(m-2)...(m-n+1)(ax+b)^{m-n} a^n$

Note :

If $y = (ax + b)^m$ then $y_n = m(m-1)(m-2)\ldots(m-n+1)(ax+b)^{m-n} a^n$.

COROLLARY

If $f(x) = (ax + b)^m$, $m \in Z$, $m > 0$, $n \in N$ then

(i) $m < n \Rightarrow f^{(n)}(x) = 0$,

(ii) $m = n \Rightarrow f^{(n)}(x) = n!\, a^n$

(iii) $m > n \Rightarrow f^{(n)}(x) = \frac{m!}{(m-n)!}(ax+b)^{m-n} a^n$.

COROLLARY

If f(x) is a polynomial function of degree less than n where $n \in N$ then $f^{(n)}(x) = 0$.

THEOREM

If $f(x) = \frac{1}{ax+b}$ then $f^{(n)}(x) = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$.**(i.e.,** If $y = \frac{1}{ax+b} \Rightarrow y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$)

THEOREM

If f(x) = log |ax + b| and $n \in N$ then $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)! a^n}{(ax+b)^n}$.

i.e., y = log |ax+b| $\Rightarrow y_n = \frac{(-1)^{n-1}(n-1)! a^n}{(ax+b)^n}$

THEOREM

If f(x) = sin(ax + b) and n ∈ N then $f^{(n)}(x) = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$.

THEOREM

If f(x) = cos(ax + b) and n ∈ N then $f^{(n)}(x) = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$.

THEOREM

If $f(x) = e^{ax+b}$ and n ∈ N then $f^{(n)}(x) = a^n e^{ax+b}$.

THEOREM

If $f(x) = c^{ax+b}$, c > 0 and n ∈ N then $f^{(n)}(x) = a^n c^{ax+b} (\log c)^n$.

THEOREM

If f(x) = $e^{ax}\sin(bx + c)$ and n ∈ N then $f^{(n)}(x) = r^n e^{ax} \sin(bx + c + n\theta)$ where a = r cos θ, b = r sin θ and $r = \sqrt{a^2 + b^2}, \theta = \text{Tan}^{-1}\left(\frac{b}{a}\right)$

THEOREM

If f(x) = $e^{ax}\cos(bx + c)$ and n∈ N then $f^{(n)}(x) = r^n e^{ax} \cos(bx + c + n\theta)$ where a = r cos θ, b = r sin θ and $r = \sqrt{a^2 + b^2}, \theta = \text{Tan}^{-1}\left(\frac{b}{a}\right)$.

Note:

If f, g are two functions in x having their n^{th} derivatives then

$(f \pm g)^{(n)}(x) = f^{(n)}(x) \pm g^{(n)}(x)$.

Note:

If f is a function in x having nth derivative and k ∈ R then $(kf)^{(n)}(x) = kf^{(n)}(x)$.

EXERCISE – 7 (a)

1. Find the nth derivative of $\sin^3 x$.

Sol: we know that $\sin 3x = 3\sin x - 4\sin^3 x \Rightarrow \sin^3 x = \frac{3\sin x - \sin 3x}{4}$

Differentiate n times w.r.t x, $\frac{d^n}{dx^n}(\sin^3 x) = \frac{1}{4}\frac{d^n}{dx^n}(3\sin x - \sin 3x)$

$= \frac{1}{4}\left[-3^n.\sin\left(3x + \frac{n\pi}{2}\right) + 3\sin\left(x + \frac{n\pi}{2}\right)\right] n \in z$

2. Find the nth derivative of sin 5x. sin 3x.?

Sol: let y $= \sin 5x.\sin 3x = \frac{1}{2}(2\sin 5x.\sin 3x)$

$\Rightarrow$ y= $\frac{1}{2}(\cos 2x - \cos 8x)$

$\Rightarrow y = \frac{1}{2}(\cos 2x - \cos 8x)$

Differentiate n times w.r.t x,

$y_n = \frac{1}{2}\frac{d^n}{dx^n}(\cos 2x - \cos 8x) \Rightarrow y_n = \frac{1}{2}\left[2^n \cos\left(2x + \frac{n\pi}{2}\right) - 8^n.\cos\left(8x + \frac{n\pi}{2}\right)\right] n \in z$

3. Find nth derivative of $e^x.\cos x.\cos 2x$

Sol: $\cos x.\cos 2x = \frac{1}{2}(2\cos 2x.\cos x) = \frac{1}{2}(\cos 3x + \cos x)$

Let $y = \frac{e^x}{2}(\cos 3x + \cos x)$

Differentiate n times w.r.t x,

$y_n = \frac{1}{2}\frac{d^n}{dx^n}(e^x \cos 3x + e^x \cos x)$

$y_n = \frac{e^x}{2}\left[(\sqrt{10})^n \cos(3x + n\tan^{-1} 3)^n + (\sqrt{2})^n + \cos(x + n\tan^{-1} 1)\right] n \in z = \frac{e^x}{2}\left[10^{n/2} \cos(3x + n\tan^{-1} 3) + 2^{n/2} \cos\left(x + \frac{n\pi}{4}\right)\right]$

4. If $y = \frac{2}{(x-1)(x-2)}$ **find** y_n

Sol: Given $y = \frac{2}{(x-1)(x-2)} = \left[\frac{1}{x-2} - \frac{1}{x-1}\right]$(partial fractions)

Differentiate n times w.r.t x,

$y_n = 2\left[\frac{(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x-1)^{n+1}}\right] = 2(-1)^n n!\left[\frac{1}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right]$

5. If $y=\frac{2x+1}{x^2-4}$**, find** y_n

Sol: Let $\frac{2x+1}{x^2-4}=\frac{A}{x-2}+\frac{B}{x+2}$

$2x+1=A(x+2)+B(x-2)$ -----(1)

In (1) ,Put $x=2\Rightarrow 5=A(4)\Rightarrow A=\frac{5}{4}$

In (1) , $x=-2\Rightarrow -3=B(-4)\Rightarrow B=\frac{3}{4}$

Therefore, $y=\frac{2x+1}{x^2-4}=\frac{5}{4(x-2)}+\frac{3}{4(x+2)}$

Differentiate n times w.r.t. x,

$$y_n=\frac{d^n}{dx^n}\left(\frac{5}{4(x-2)}+\frac{3}{4(x+2)}\right)$$

$$y_n=\frac{5}{4}\left[\frac{(-1)^n n!}{(x-2)^{n+1}}\right]+\frac{3}{4}\frac{(-1)^n n!}{(x+2)^{n+1}}=\frac{(-1)^n n!}{4}\left(\frac{5}{(x-2)^{n+1}}+\frac{3}{(x+2)^{n+1}}\right)$$

1. Find the nth derivative of (i) $\frac{x}{(x-1)^2(x+1)}$ **(ii)** $\frac{1}{(x-1)(x+2)^2}$ **(iii)** $\frac{x^3}{(x-1)(x+1)}$

(iv) $\frac{x}{x^2+x+1}$ **(v)** $\frac{x+1}{x^2-4}$ **(vi) Log** $(4x^2-9)$

Sol: i)

Let $y=\frac{x}{(x-1)^2(x+1)}$

Resolving into partial fractions

$$\frac{x}{(x-1)^2(x+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$$

$x=A(x-1)(x+1)+B(x+1)+C(x-1)^2$ ----- (1

In (1), put $x=1\Rightarrow 1=B(1+1)=2B\Rightarrow B=\frac{1}{2}$

In (1), put $x=-1\Rightarrow -1=C(-1-1)^2=4C\Rightarrow C=-\frac{1}{4}$

Equating the co . efficient of $x^2\Rightarrow A+B=0\Rightarrow A=-\frac{1}{2}$

Therefore, $y=-\frac{1}{2(x-1)}+\frac{1}{2(x-1)^2}-\frac{1}{4(x+1)}$

Differentiate n times w.r.t. x,

$$y_n=\frac{d^n}{dx^n}\left(-\frac{1}{2(x-1)}+\frac{1}{2(x-1)^2}-\frac{1}{4(x+1)}\right)$$

$$y_n = \frac{(-1)^n n!}{2(x-1)^{n+1}} + + \frac{1}{2}\frac{(-2)(-3).....(-2-n+1)}{(x-1)^{n+2}} - \frac{1}{4}\frac{(-1)^n n!}{(x+1)^{n+1}}$$

$$= \frac{(-1)^n n!}{2.(x-1)^{n+1}} + \frac{(-1)^n (n+1)!}{2(x-1)^{n+2}} - \frac{1}{4}\frac{(-1)^n n!}{(x+1)^{n+1}}$$

$$= (-1)^n n!\left[\frac{1}{2(x-1)^{n+1}} + \frac{n+1}{2(x-1)^{n+2}} - \frac{1}{4(x+1)^{n+1}}\right]$$

(ii) $y = \frac{1}{(x-1)(x+2)^2}$

Resolving into partial fractions $\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

$1 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$ ---(1)

In (1) put $x = 1 \Rightarrow 1 = A(1+2)^2 = 9A \Rightarrow A = \frac{1}{9}$

In (1) put $x = -2 \Rightarrow 1 = C(-2-1) = -3C \Rightarrow C = -\frac{1}{3}$

Equating the co – efficient of x^2 In (1)

$A + B = 0 \Rightarrow B = -A = -\frac{1}{9}$

$\therefore\ y = \frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2}$

Differentiate n times w.r.t. x ,

$$y_n = \frac{d^n}{dx^n}\left(\frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2}\right)$$

$$y_n = \frac{(-1)^n n!}{9(x-1)^{n+1}} - \frac{(-1)^n n!}{9(x+2)^{n+1}} - \frac{1}{3}\frac{(-1)^n (n+1)}{(x+2)^{n+2}} = (-1)^n n!\left[\frac{1}{9(x-1)^{n+1}} - \frac{1}{9(x+2)^{n+1}} - \frac{n+1}{3(x+2)^{n+2}}\right]$$

(iii) $y = \frac{x^3}{(x-1)(x+1)}$

Ans: $\frac{(-1)^n n!}{2}\left[\frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}}\right]$

(iv) $\frac{x}{x^2 + x + 1}$

Ans: $y_n = \frac{(-1)^n n!}{r^{n+1}}\left[\cos(n+1)\theta - \frac{1}{\sqrt{3}}\sin(n+1)\theta\right]$

(v) $y = \frac{x+1}{x^2 - 4}$

Ans: $\frac{(-1)^n n!}{4}\left[\frac{3}{(x-2)^{n+1}} + \frac{1}{(x+2)^{n+1}}\right]$

(vi) $y=\log\left(4x^2-9\right)$

Given $y=\log\left(4x^2-9\right)=\log[2x-3][2x+3]$

$=\log(2x-3)+\log(2x+3)$

Differentiating n times,

$$y_n=\frac{d^n}{dx^n}\left(\log(2x-3)+\log(2x+3)\right)$$

$$y_n=\frac{(-1)^{n-1}2^n(n-1)!}{(2x-3)^n}+\frac{(-1)^{n-1}2^n(n-1)!}{(2x+3)^n}$$

$$=(-1)^{n-1}2^n(n-1)!\left[\frac{1}{(2x-3)^n}+\frac{1}{(2x+3)^n}\right]$$

2. If y = $\frac{a+bx}{c+dx}$ then show that $2y_1y_3=3y\frac{2}{2}$

Sol: Given $y=\frac{a+bx}{c+dx}$

Differentiate w.r.t.x ,

$$\frac{dy}{dx}=\frac{(c+dx)b-(a+bx).d}{(c+dx)^2}$$

$$\Rightarrow y_1=\frac{bc+bdx-ad-bdx}{(c+dx)^2}=\frac{bc-ad}{(c+dx)^2}$$

Again diff. w.t.t x,

$$y_2=\frac{(bc-ad)(-2)d}{(c+dx)^3}=\frac{-2d(bc-ad)}{(c+dx)^3}$$

Diff.wrt.x, we get

$$y_3=\frac{-2d(bc-ad)(-3).d}{(c+dx)^4}=\frac{6d^2(bc-ad)}{(c+dx)^4}$$

L.H.S.= $2y_1y_3=\frac{2(bc-ad)}{(c+dx)^2}.\frac{6d^2(bc-ad)}{(c+dx)^4}=\frac{12d^2(bc-ad)^2}{(c+dx)^6}$

$=3\left[\frac{-2d(bc-ad)}{(c+dx)^3}\right]^2=3y_2^2$ =R.H.S.

3. If y = sin (sinx), then show that $y_2+(\tan x)y_1+y\cos^2 x=0$

Sol: Given y = sin (sinx)

Diff. wrt x,

$y_1=\cos(\sin x)\cos x$

Diff. wrt x,

$y_2=\cos x[-\sin(\sin x)]\cos x-\cos(\sin x)\sin x$

$=-\cos^2 x.\sin(\sin x)-\sin x.\cos(\sin x)$

LHS= $y_2+(\tan x)y_1+y\cos^2 x$

$=-\cos^2 x\sin(\sin x)-\sin x.\cos(\sin x)+\frac{\sin x}{\cos x}\cos x(\sin x)+\sin x.\cos(\sin x)=0$ = RHS.

4. If $y = ax^{n+1} + bx^{-n}$**, then show that** $x^2 y_2 = n(n+1)y$**.**

Sol: $y = ax^{n+1} + bx^{-n}$

Diff. wrt. X,

$y_1 = a(n+1)x^n - bnx^{-(n+1)}$

Diff. wrt x, $\Rightarrow y_2 = a.n(n+1)x^{n-1} + bn.(n+1)x^{-(n+2)}$

$\Rightarrow x^2.y_2 = n(n+1).x^2\left[a.x^{n-1} + b.x^{-n-2}\right]$

$= n(n+1)\left(ax^{n+1} + bx^{-n}\right) = n(n+1)y$

5. If $y = ae^{nx} + be^{-nx}$**, then show that** $y_2 = n^2 y$

Sol: $y = ae^{nx} + be^{-nx}$

$\Rightarrow y_1 = a.n.e^{nx} - bn.e^{-nx}$

$\Rightarrow y_2 = an^2e^{nx} + bn^2.e^{-nx} = n^2\left(ae^{nx} + be^{-nx}\right)$

$\Rightarrow y_2 = n^2 y$

6. If $y = e^{\frac{-kx}{2}}(a\cos nx + b\sin nx)$ **then show that** $y_2 + ky_1 + \left(n^2 + \frac{k^2}{4}\right)y = 0$.

Sol. $y = e^{\frac{-kx}{2}}(a\cos nx + b\sin nx)$

Differentiating w.r.to x.

$\Rightarrow y_1 = e^{\frac{-kx}{2}}[-an\sin nx + bn\cos nx] + \left(-\frac{k}{2}\right)e^{\frac{-lx}{2}}[a\cos nx + b\sin nx]$

$\Rightarrow y_1 = +e^{\frac{-kx}{2}}n(-a\sin nx + b\cos nx) - \frac{k}{2}y$

$\Rightarrow y_1 + \frac{k}{2}y = +n.e^{\frac{-kx}{2}}(-a\sin nx + b\cos nx) - (1)$

Differentiating w.r.to x.

$y_2 + \frac{k}{2}y_1 = +n.e^{\frac{-kx}{2}}\left(-\frac{k}{2}\right)(-a\sin nx + b\cos nx) + n.e^{\frac{-kx}{2}}[-an.conx - bn\sin nx]$

$= -\frac{k}{2}\left(y_1 + \frac{k}{2}y\right) - n^2y = -\frac{k}{2}y_1 - \frac{k^2}{4}y - n^2y$

$\therefore y_2 + ky_1 + \left(n^2 + \frac{k^2}{4}\right)y = 0$

7. If f(x) = (x-a)^2 $\phi(x)$ where ϕ is a polynomial with rational co-efficient, then show that f(a) = 0 = $f^1(a)$ **and** $f''(a) = 2\phi(a)$

Sol: Given $f(x) = (x-a)^2 \phi(x)$

Diff. wrt. X,

$f'(x) = (x-a)^2 \phi'(x) + 2(x-a)\phi(x)$

Diff.wrt.x,

$f''(x) = (x-a)^2 \phi''(x) + 2(x-a)\phi'(x) + 2(x-a)\phi'(x) + 2\phi(x) = (x-a)^2 \phi''(x) + 4(x-a)\phi'(x) + 2\phi(x)$

Now $f(a) = (a-a)^2 \phi(a) = 0.\phi(a) = 0$

And $f'(a) = 0.\phi'(a) + 0.\phi(a) = 0 + 0 = 0$

$\therefore f(a) = 0 = f''(a)$

$f''(a) = 0.\phi''(a) + 4.0\phi'(a) + 2\phi(a) = 2\phi(a)$.

8. If y = $e^x.\cos x$**, then show that** $y_4 + 4y = 0$

Sol: $y = e^x.\cos x$

Here a =1 , b=1 and n=4

$y = e^{ax}\cos(bx+c) \Rightarrow y_n = \left(\sqrt{a^2+b^2}\right)^n e^{ax}\cos(bx+c+n\theta)$ where $\theta = \text{Tan}^{-1}\left(\frac{b}{a}\right)$.

Now $y_4 = \left(\sqrt{1^2+1^2}\right)^4 e^x \cos(x + 4\tan^{-1}\frac{1}{1})$

$y_4 = \left(\sqrt{2}\right)^4 e^x \cos(x + 4\tan^{-1} 1)$

$y_4 = \left(\sqrt{2}\right)^4 e^x \cos(x + 4\frac{\pi}{4}) \Rightarrow y_4 = -4e^x \cos x$

$y_4 = -4y \Rightarrow y_4 + 4y = 0$

9. If y = x + tan x , then show that $y_2 \cos^2 x + 2x = 2y$

10. If $y\sqrt{1+x^2} = \log\left(x+\sqrt{1+x^2}\right)$**, then show that** $\left(1+x^2\right)y_1 + xy = 1$

Sol: $y\sqrt{1+x^2} = \log\left(x+\sqrt{1+x^2}\right)$

Differentiating w.r.to x ,

$$y\frac{1}{2\sqrt{1+x^2}}2x + \sqrt{1+x^2}.y_1$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{1}{2\sqrt{1+x}}2x\right)$$

$$\Rightarrow \left(1+x^2\right)y_1 + xy \;=\; \sqrt{1+x^2}.\frac{1}{x+\sqrt{1+x^2}}.\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} = 1$$

11. If y = a cos x + (b + 2x) sin x, then show that $y_2 + y = 4\cos x$

Sol: y = a cosx + (b + 2x) sinx ______(1)

Differentiating w.r.to x

$$y_1 = -a\sin x + (b+2x).\cos x + 2\sin x$$

Again differentiating w.r.to x

$$y_2 = -a\cos x + 2\cos x - (b+2x)\ \sin x + 2\cos x$$

$$= 4\cos x - \left[a\cos x + (b+2x).\sin x\right]$$

=4 cosx – y (from (1))

Hence $y_2 + y = 4\cos x$

12. If y = $a + be^{-4x}$ **, then show that** $y_2 + 4y_1 = 0$

13. If y = ax + b log x, then show that $\left(x^2\log x - x^2\right)y_2 - xy_1 + y = 0$

Sol: Given y = ax + blogx

Differentiating wrt. X,

$$y_1 = a + \frac{b}{x},$$

Diff. wrt.x , $\Rightarrow y_2 = -\frac{b}{x^2}$

Now L.H.S = $\left(x^2 \log x - x^2\right) y_2 - xy_1 + y$

$= x^2 (\log x - 1)\left(-\frac{b}{x^2}\right) - x\left(a + \frac{b}{x}\right) + ax + b\log x$

$= -b(\log x - 1) - ax - b + ax + b\log x$

= - b log x +b – b log x = 0 =R.H.S

14. If y = a cosec (b – x), then show that $yy_2 - 2y_1^2 = y^2$

Sol: y = a cosec (b – x) -----(1)

Differentiating w.r.to x

$y_1 = -a\operatorname{cosec}(b-x)\cot(b-x)(-1)$

y_1 =y.cot(b – x) ----(2)

Diff .with respect x,

$y_2 = -y\operatorname{cosec}^2(b-x)(-1) + \cot(b-x).y_1$

$= y\operatorname{cosec}^2(b-x) + y_1\cot(b-x)$

$\Rightarrow y_2 = y\operatorname{cosec}^2(b-x) + \frac{y_1^2}{y}$

$\Rightarrow yy_2 = y^2\operatorname{cosec}^2(b-x) + y_1^2$

$\Rightarrow yy_2 = 2y_1^2\operatorname{cosec}^2(b-x) - y^2\cot^2(b-x) = y^2$

15. If $ay^4 = (x+b)^5$, then show that $5yy_2 = y_1^2$.

Sol: Given $ay^4 = (x+b)^5$

$\Rightarrow y^4 = \frac{1}{a}(x+b)^5$

$\Rightarrow y = \frac{1}{a^{1/4}}(x+b)^{5/4}$ (finding the 4th root)

Differentiating w.r.to x, $\Rightarrow y_1 = \frac{1}{a^{1/4}}.\frac{5}{4}.(x+b)^{1/4}$

Diff.wrt.x, $\Rightarrow y_2 = \frac{1}{a^{1/4}}.\frac{5}{4}.\frac{1}{4}(x+b)^{-3/4}$

Now

$$\text{L.H.S} = 5yy_2 = 5\frac{1}{a^{1/4}}(x+b)^{5/4}\frac{5}{16a^{1/4}}(x+b)^{-3/4} = \frac{25}{16\left(a^{1/4}\right)^2}(x+b)^{2/4}$$

$$= \left[\frac{5}{4a^{1/4}}(x+b)^{1/4}\right]^2 = y_1^2$$

16. If y = 6(x+1) +(A+Bx)e^{3x}, then show that $y_2 - 6y_1 + 9y = 54x + 18$

III.

1. If $\theta \in [-\pi, \pi]$ **be such that** $\cos\theta = \frac{x}{\sqrt{x^2+1}}$ **and** $\sin\theta = \frac{1}{\sqrt{1+x^2}} \forall x \in R,$ **then prove that**

i) nth derivative of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ **is** $\frac{2(-1)^{1-1}(n-1)!}{\left(1+x^2\right)^{n/2}}\sin n\theta$

ii) nth derivative of $\tan^{-1}\left(\frac{1+x}{1-x}\right)$ **is** $(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$

iii) nth derivative of $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ **is** $3(-1)^{n-1}.(n-1)!\sin^n\theta(\sin n\theta)$

Sol: i) let $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, put x = tan θ then $\theta = \tan^{-1}x$

$$\Rightarrow y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta = 2\tan^{-1}x$$

Therefore, $y = 2\tan^{-1}x$

Differentiate wrt. X,

$$\Rightarrow y_1 = 2\left(\frac{1}{1+x^2}\right) = \frac{2}{1+x^2}$$

$$\Rightarrow y_1 = \frac{1}{i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right] \quad (\text{ partial fractions})$$

Differentiating n-1 times w.rt. x,

$$y_n = D^{n-1}\left[\frac{1}{i}\left(\frac{1}{x-i}-\frac{1}{x+i}\right)\right]$$

$$=\frac{1}{i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-i)^n}-\frac{(-1)^{n-1}(n-1)!}{(x+i)^n}\right]\_\_\_(1)$$

let $x = r\cos\theta$ and $1=r\sin\theta \Rightarrow \sin\theta = \frac{1}{r}$

Then x + i = r $(\cos\theta + i\sin\theta)$

Now $1+x^2 = r^2 \Rightarrow r = \sqrt{1+x^2}$

and $\tan\theta = \frac{1}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{x}\right)$

Now $(x+i)^n = r^n(\cos n\theta + i\sin n\theta)$

$$\Rightarrow \frac{1}{(x+i)^n} = r^{-n}(\cos n\theta - i\sin n\theta)$$

$$(x-i)^n = r^n(\cos n\theta - i\sin n\theta)\ \frac{1}{(x-i)^n} = r^{-n}(\cos n\theta + i\sin n\theta)$$

$$\therefore \frac{1}{(x-i)^n}-\frac{1}{(x+i)^n} = r^{-n}(2i\sin n\theta)\_\_\_(2)$$

From (1) and (2) $y_n = \frac{1}{i}\left[(-1)^{n-1}(n-1)!\left\{r^{-n}(2i\sin n\theta)\right\}\right]$

$$= 2(-1)^{n-1}(n-1)!\frac{1}{r^n}\sin n\theta$$

$$= 2(-1)^{n-1}(n-1)!\sin n\theta.\sin^n(\theta)$$

$$\because \frac{1}{r} = \sin\theta$$

$$= 2(-1)^{n-1}(n-1)!\sin n\theta\left(\frac{1}{\sqrt{1+x^2}}\right)^n = 2(-1)^{n-1}(n-1)!\frac{\sin n\theta}{(1+x^2)^{n/2}}, \theta = \tan^{-1}\left(\frac{1}{x}\right)$$

ii) Let $y = \tan^{-1}\left(\frac{1+x}{1-x}\right) = \tan^{-1}(1) + \tan^{-1}(x)$ proceed as above problem

iii) Let $y = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right) = 3\tan^{-1}(x)$ as above problem.

2. If $ax^2 + 2hxy + by^2 = 1$, then show that $\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx+by)^3}$

Sol: Given equation

$ax^2 + 2hxy + by^2 = 1$ ________(1)

Differentiating w.r.to x,

$$a(2x) + 2h\left(x.\frac{dy}{dx} + y.1\right) + b.2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2\frac{dy}{dx}(hx+by) = -2(ax+hy)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(ax+hy)}{(hx+by)} \text{ ________(2)}$$

Differentiating w.r.to x

$$\frac{d^2y}{dx^2} = \frac{\left[(hx+by)\left(a + h\frac{dy}{dx}\right) - (ax+hy)\left(h + b.\frac{dy}{dx}\right)\right]}{(hx+by)^2}$$

$$= \frac{-(hx+by)\left[a - \frac{h(ax+hy)}{hx+by}\right] + (ax+hy)\left[h - \frac{b(ax+hy)}{hx+by}\right]}{(hx+by)^2}$$

$$= \frac{(hx+by)(ahx+aby-ahx-h^2y) + (ax+hy)(h^2x+bhy-ahx-bhy)}{(hx+by)^3}$$

$$= \frac{(hx+by)(ab-h^2)y + (ax+hy)(h^2-ah)x}{(hx+by)^3}$$

$$= \frac{(h^2-ab)(hxy+by^2+ax^2+hxy)}{(hx+by)^3}$$

$$= \frac{(h^2-ab)(ax^2+2hxy+by^2)}{(hx+by)^3} = \frac{(h^2-ab).1}{(hx+by)^3} = \frac{(h^2-ab)}{(hx+by)^3}$$

3. If $y = ae^{-bx}\cos(cx+d)$ **then show that** $y_2 + 2by_1 + (b^2+c^2)y = 0$

Sol: $y = ae^{-bx}\cos(cx+d)$---------(1)

Diff. wr.t x,

$y_1 = a.e^{-bx}[-\sin(cx+d)c] + a\cos(cx+d)e^{-bx}(-b)$

$= (-b).y - ac.e^{-bx}.\sin(cx+d)$

$y_1 + by = -ac.e^{-bx}\sin(cx+d)$______(2)

Differentiating w.r.to x

$y_2 + by_1 = -ac(e^{-bx}\cos(cx+d)).c + (-b)e^{-bx}.\sin(cx+d))$

$= -c^2(a.e^{-bx}\cos(cx+d)) - b[-ac.e^{-bx}\sin(cx+d)]$

$= -c^2[a.e^{-bx}\cos(cx+d)] - b[-ae.e^{-bx}\sin(cx+d)]$

$= -c^2y - b(y_1+by)$ from (1) and (2)

$= -c^2y - by_1 - b^2y$

$y_2 + 2by_1 + (b^2+c^2)y = 0$.

4. **if** $y = \frac{d^n}{dx^n}(x^n\log x)$, **prove that** $y_n = n!\left[\log x + 1 + \frac{1}{2} + \frac{1}{3}.......+\frac{1}{n}\right](x>0)$

Sol.

$$y_n = \frac{d^n}{dx^n}(x^n.\log x)$$
$$= D^{n-1}(Dx^n.\log x)$$
$$= D^{n-1}\left(x^n\frac{1}{x} + nx^{n-1}\log x\right)$$
$$= D^{n-1}(x^{n-1} + nx^{n-1}\log x)$$
$$= D^{n-1}x^{n-1} + nD^{n-1}x^{n-1}\log x$$
$$= (n-1)! + n.y_{n-1}$$
$$\rightarrow y_n - ny_{n-1} = (n-1)!$$

dividing with n!,

$$\frac{y_n}{n!}-\frac{y_{n-1}}{(n-1)!}=\frac{1}{n}\text{-------(1)}$$

In (1), put n=1,2,3,4,5-----n, then

$$\frac{y_1}{1!}-\frac{y_0}{0!}=1$$

$$\frac{y_2}{2!}-\frac{y_1}{1!}=\frac{1}{2}$$

……………

$$\frac{y_{n-1}}{(n-2)!}-\frac{y_{n-2}}{(n-2)}=\frac{1}{(n-1)}$$

$$\frac{y_n}{n!}-\frac{y_{n-1}}{(n-1)!}=\frac{1}{n}$$

Adding above n equations, we get

$$\frac{y_n}{n!}-\frac{y_0}{1!}=1+\frac{1}{2}+.....+\frac{1}{n}$$

$$\Rightarrow \frac{y_n}{n!}=\frac{y_0}{1!}+1+\frac{1}{2}+.....+\frac{1}{n}$$

$$\Rightarrow \frac{y_n}{n!}=\frac{y_0}{1!}+1+\frac{1}{2}+.....+\frac{1}{n}$$

$$\Rightarrow y_n=n!\left(\frac{y_0}{1!}+1+\frac{1}{2}+.....+\frac{1}{n}\right)$$

$$y_n=n!\left(\log x+1+\frac{1}{2}+.....+\frac{1}{n}\right),\ \ here\ \mathrm{y}_0=D^0x^0\log x=\log x$$

LEIBNITZ THEOREM

If f, g are two functions in x having n^{th} derivatives then

$$(fg)^{(n)}(x) = {}^nC_0 f^{(n)}(x)g(x) + {}^nC_1 f^{(n-1)}(x)g'(x) + {}^nC_2 f^{(n-2)}(x)g^{(2)}(x) + \ldots + {}^nC_r f^{(n-r)}(x)g^{(r)}(x) + \ldots + {}^nC_n f(x)g^{(n)}(x).$$

Proof :

Let S(n) be the statement that

$$(fg)^n(x) = {}^nC_0 f^n(x)g(x) + {}^nC_1 f^{n-1}(x)g'(x) + \ldots + {}^nC_r f^{(n-r)}(x)g^r(x) + \ldots + {}^nC_n f(x)g^n(x)$$

Now $(fg)'(x) = f(x)g'(x) + g(x)f'(x)$ (product rule)

From above statement, $(fg)^1 = {}^1C_0 f'(x)g(x) + {}^1C_1 f(x)g'(x)$

$\therefore$ S(1) is true.

Assume that S(k) is true.

$$\therefore (fg)^{(k)}(x) = {}^kC_0 f^{(k)}(x)g(x) + {}^kC_2 f^{(k-1)}(x)g'(x) + \ldots + {}^kC_r f^{(k-r)}(x)g^{(r)}(x) + \ldots + {}^kC_k f(x)g^{(k)}(x)$$

Now

$$(fg)^{(k+1)}(x) = [(fg)^{(k)}(x)]'$$

$$= \frac{d}{dx}\left({}^kC_0 f^{(k)}(x)g(x) + {}^kC_2 f^{(k-1)}(x)g'(x) + \ldots + {}^kC_r f^{(k-r)}(x)g^{(r)}(x) + \ldots + {}^kC_k f(x)g^{(k)}(x)\right)$$

$$= {}^kC_0 f^{(k+1)}(x)g(x) + {}^kC_0 f^{(k)}(x)g'(x) + {}^kC_1 f^{(k)}(x)g'(x) + {}^kC_1 f^{(k-1)}(x)g^{(2)}(x) + \ldots + {}^kC_r f^{(k-r+1)}(x)g^{(r)}(x) + {}^kC_r f^{(k-r)}(x)g^{(r+1)}(x) + \ldots + {}^kC_k f'(x)g^{(k)}(x) + {}^kC_k f(x)g^{(k+1)}(x)$$

$$= f^{(k+1)}(x)g(x) + \left[{}^kC_0 + {}^kC_1\right]f^{(k)}(x)g'(x) + \left[{}^kC_1 + {}^kC_2\right]f^{(k-1)}(x)g^{(2)}(x) + \ldots + \left[{}^kC_{r-1} + {}^kC_r\right] f^{(k-r+1)}(x)g^{(r)}(x) + \ldots + f(x)g^{(k+1)}(x)$$

$$= {}^{(k+1)}C_0 f^{(k+1)}(x)g(x) + {}^{(k+1)}C_1 f^{(k+1-1)}(x)g'(x) + {}^{(k+1)}C_2 f^{(k+1-2)}(x)g^{(2)}(x) + \ldots + {}^{(k+1)}C_r f^{(k+1-r)}(x) g^{(r)}(x) + \ldots + {}^{(k+1)}C_{k+1} f(x)g^{(k+1)}(x)$$

$\therefore$ S(k + 1) is true.

By principle of Mathematical Induction S(n) is true for all $n \in$ N.

$$\therefore (fg)^{(n)}(x) = {}^nC_0 f^{(n)}(x)g(x) + {}^nC_1 f^{(n-1)}(x)g'(x) + \ldots + {}^nC_r f^{(n-r)}(x)g^{(2)}(x) + \ldots + {}^nC_n f(x)g^{(n)}(x)$$