The Leibnitz Formula for the n'th Derivative of a Product

Theorem 1. *Let $u(x)$ and $v(x)$ be functions of class C^n, i.e. functions with continuous n'th derivative. Then their product is also of class C^n, and*

$$\frac{d^n}{dx^n}\left[u(x)v(x)\right] = \sum_{r=0}^{n} \binom{n}{r} \frac{d^r}{dx^r}\left[u(x)\right] \frac{d^{n-r}}{dx^{n-r}}\left[v(x)\right],$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is the usual binomial coefficient.

Proof. This proceeds by induction on n, the result being trivial for $n = 0$ and reducing for $n = 1$ to the well-known rule for differentiating a product (once). Suppose then that

$$\frac{d^{n-1}}{dx^{n-1}}\left[uv\right] = \sum_{r=0}^{n-1} \binom{n-1}{r} u^{(r)} v^{(n-1-r)} \qquad \text{for some } n \geq 1,$$

where $u^{(r)} = \frac{d^r u}{dx^r}$, $v^{(n-1-r)} = \frac{d^{n-1-r} v}{dx^{n-1-r}}$. Then

$$\begin{aligned}
\frac{d^n}{dx^n}(uv) &= \sum_{r=0}^{n-1} \binom{n-1}{r} \left[u^{(r)} v^{(n-r)} + u^{(r+1)} v^{(n-1-r)}\right] \\
&= uv^{(n)} + \sum_{r=1}^{n-1} \binom{n-1}{r} u^{(r)} v^{(n-r)} + \sum_{r=0}^{n-2} \binom{n-1}{r} u^{(r+1)} v^{(n-1-r)} + u^{(n)} v \\
&= uv^{(n)} + \sum_{r=1}^{n-1} \binom{n-1}{r} u^{(r)} v^{(n-r)} + \sum_{r=1}^{n-1} \binom{n-1}{r-1} u^{(r)} v^{(n-r)} + u^{(n)} v \\
&= uv^{(n)} + \sum_{r=1}^{n-1} \left[\binom{n-1}{r} + \binom{n-1}{r-1}\right] u^{(r)} v^{(n-r)} + u^{(n)} v,
\end{aligned}$$

where

$$\binom{n-1}{r} + \binom{n-1}{r-1} = \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-r)!} = \frac{(n-1)!}{r!(n-r)!}\left[(n-r)+r\right] = \binom{n}{r},$$

so

$$\frac{d^n}{dx^n}(uv) = uv^{(n)} + \sum_{r=1}^{n-1} \binom{n}{r} u^{(r)} v^{(n-r)} + u^{(n)} v = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)}.$$

This completes the proof by induction on n. ∎