

Lesson 10

SUCCESSIVE DIFFERENTIATION: LEIBNITZ'S THEOREM

OBJECTIVES

At the end of this session, you will be able to understand:

- Definition
- n^{th} Differential Coefficient of Standard Functions
- Leibnitz's Theorem

DIFFERENTIATION: If $y = f(x)$ be a differentiable function of x , then $\frac{dy}{dx} = f'(x)$ is called the first differential coefficient of y w.r.t x .

Hence, differentiating both side w.r.t. x , we have

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \left(\frac{d}{dx} \right) [f'(x)] = f''(x).$$

Let $\left(\frac{d}{dx} \right) \left(\frac{dy}{dx} \right)$ be represented by $\frac{d^2y}{dx^2}$; then $\frac{d^2y}{dx^2} = f''(x)$

Similarly $\left(\frac{d}{dx} \right) \left(\frac{d^2y}{dx^2} \right)$ is represented by $\frac{d^3y}{dx^3}$; ie $\frac{d^3y}{dx^3} = f'''(x)$ and so on

The expressions $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{d^ny}{dx^n}$ are called the first, second, third,nth differential coefficient of y .

These function are usually written as

$y', y'', y''', \dots, y^{(n)}$ or $y_1, y_2, y_3, \dots, y_n$ and also $Dy, D^2y, D^3y, \dots, D^ny$.

n^{th} DIFFERENTIAL COEFFICIENT OF STANDARD FUNCTION:

(i) **Differential Coefficient of x^m :**

If $y = x^m$, then $y_1 = mx^{m-1}$;

$y_2 = m(m-1)x^{m-2}$; $y_3 = m(m-1)(m-2)x^{m-3}$ and so on

In general, $y_n = m(m-1)(m-2)(m-3)\dots(m-n+1)x^{m-n}$;

Note: If m be a positive integer, we have

$$y_n = 1.2.3.....m = m!;$$

$$\text{Hence } D_n(x^m) = m(m-1)(m-2)(m-3).....(m-n+1)x^{m-n}$$

(ii) Differential Coefficient of $(ax + b)^m$:

$$\text{If } y = (ax+b)^m, \text{ then } y_1 = am(ax + b)^{m-1};$$

$$y_2 = a^2 m(m-1)(ax + b)x^{m-2}; y_3 = a^3 m(m-1)(m-2)(ax + b)x^{m-3} \text{ and so on.}$$

$$\text{In general } y_n = a^n m(m-1)(m-2)(m-3).....(m-n+1)(ax + b)x^{m-n}$$

[Note: In the first differentiation, the last term in it is $(m-1+1)$; in the second differentiation it is $(m-1)$ i.e. $(m-2+1)$; in the third differentiation it is $(m-2)$ i.e. $(m-3+1)$. So the n th differentiation it will be $(m-n+1)$.]

$$\text{Hence } D(ax + b)^m = a^n m(m-1)(m-2)(m-3).....(m-n+1)(ax + b)^{m-n}$$

$$\text{or } D(ax + b)^m = \frac{m!}{(m-n)!} a^n (ax + b)^{m-n}$$

In case m is negative integer, let $m = -p$, where p is positive integer, then

$$D(ax + b)^{-p} = a^n (-p)(-p-1)(-p-2).....[-p-(n-1)](ax + b)^{-p-n}$$

$$= a^n (-1)^n p(p+1)(p+2).....[(p-n+1)](ax + b)^{-p-n}$$

$$= (-1)^n \frac{(p-n+1)!}{(p-1)!} a^n (ax + b)^{-p-n}$$

Note1. If $m = n$, then $D^n(ax + b)^{-p} = a^n !$

Note2. If $m = -1$, we have $D^n(ax + b)^{-1} = (-1)(-1-1).....(-1-n+1)a^n (ax + b)^{-1-n}$

$$(-1)^n 1.2.3.....na^n (ax + b)^{-1-n} = (-1)^n n! a^n (ax + b)^{-1-n} = \frac{(-1)^n n! a^n}{(ax + b)^{n+1}}.$$

(iii) Differential Coefficient of $(ax + b)$:

$$\text{If } y = \log(ax + b), \text{ then } y_1 = \frac{a}{(ax + b)} = a(ax + b)^{-1} = \frac{a(0!)}{(ax + b)}$$

$$y_2 = \frac{a^2 \cdot 1}{(ax + b)^2} = -\frac{a^2 \cdot (1!)}{(ax + b)^2}; y_3 = \frac{a^3 \cdot 2}{(ax + b)^3} = -\frac{a^3 \cdot (2!)}{(ax + b)^3}.$$

$$y_4 = \frac{a^4 \cdot 2 \cdot 3}{(ax + b)^4} = (-1)^3 \cdot \frac{a^4 \cdot (3!)}{(ax + b)^4} \text{ and so on.}$$

In general, $y_n = (-1)^{n-1} \cdot \frac{a^n \cdot (n-1)!}{(ax+b)^n}$

Hence $D^n \log(ax+b) = (-1)^{n-1} \cdot \frac{a^n \cdot (n-1)!}{(ax+b)^n}$.

Note: $D^n \log x = \frac{(-1)^{n-1} (n-1)!}{x^n}$.

(iv) Differential Coefficient of a^{bx} :

If $y = a^{bx}$, then $y_1 = ba^{bx} \log a$, $y_2 = b^2 a^{bx} (\log a)^2$.

$y_3 = b^3 a^{bx} (\log a)^3$, and so on.

In general, $y_n = b^n a^{bx} (\log a)^n$,

Hence $D^n .a^{bx} = b^n a^{bx} (\log_e a)^n$.

(v) Differential Coefficient of e^{ax} :

If $y = e^{ax}$, then

$y_1 = ae^{ax}$, $y_2 = a^2 e^{ax}$, $y_3 = a^3 e^{ax}$, $y_4 = a^4 e^{ax}$ and so on.

In general, $y_n = a^n e^{ax}$, Hence $D^n e^{ax} = a^n e^{ax}$.

(vi) Differential Coefficient of $\sin(ax+b)$:

If $y = \sin(ax+b)$, then

$$y_1 = a \cos(ax+b) = a \sin \left[\frac{\pi}{2} + (ax+b) \right]$$

$$y_2 = -a^2 \cos(ax+b) = a^2 \sin \left[\frac{2\pi}{2} + (ax+b) \right], \text{ and so on.}$$

In General, $y_1 = a^n \sin \left(ax+b + \frac{1}{2} n\pi \right)$.

Hence, $D^n (ax+b) = a^n \sin \left[ax+b + \frac{1}{2} n\pi \right]$

Note: $D^n \sin x = \sin \left[x + \left(\frac{n\pi}{2} \right) \right]$

(vii) Differential Coefficient of $\cos(ax + b)$:

If $y = \cos(ax + b)$, then

$$y_1 = -a \sin(ax + b) = a \cos\left(\frac{\pi}{2} + ax + b\right)$$

$$y_2 = -a^2 \sin\left(\frac{\pi}{2} + ax + b\right) = a^2 \cos\left(\frac{2\pi}{2} + ax + b\right), \text{ and so on}$$

$$\text{In general, } y_n = a^n \cos\left(ax + b + \frac{1}{2}n\pi\right).$$

$$\text{Hence, } D^n \cos x(ax + b) = a^n \cos\left(ax + b + \frac{1}{2}n\pi\right).$$

$$\text{Note : } D^n \cos x = \cos\left(x + \frac{1}{2}n\pi\right).$$

(viii) Differential Coefficient of $e^{ax} \sin(bx + c)$ and $e^{ax} \cos(bx + c)$:

If $y = e^{ax} \sin(bx + c)$, then

$$y_1 = e^{ax} \sin(bx + c) + be^{ax} \cos(bx + c) = e^{ax} [a \sin(bx + c) + b \cos(bx + c)]$$

Putting $a = r \cos \phi$ and $b = r \sin \phi$, we get

$$y_1 = re^{ax} \sin(bx + c + \phi), \text{ where } r^2 = a^2 + b^2 \text{ and } \phi = \tan^{-1}\left(\frac{b}{a}\right),$$

similarly, $y_1 = r^2 e^{ax} \sin(bx + c + 2\phi)$, and so on.

$$\text{Hence, } D^n e^{ax} \sin(bx + c) = r^n e^{ax} \sin(bx + c + n\phi)$$

$$\text{where } r = (a^2 + b^2)^{\frac{1}{2}} \text{ and } \phi = \tan^{-1}\left(\frac{b}{a}\right).$$

$$\text{Similarly, } D^n e^{ax} \cos(bx + c) = r^n e^{ax} \cos(bx + c + n\phi)$$

$$\text{where } r = (a^2 + b^2)^{\frac{1}{2}} \text{ and } \phi = \tan^{-1}\left(\frac{b}{a}\right).$$

Example. Find the n^{th} derivative of $e^{ax} \sin bx \cos cx$.

Solution.

$$\begin{aligned} \text{Let } y &= e^{ax} \sin bx \cos cx = \frac{1}{2} e^{ax} (2 \sin bx \cos cx) \\ &= \frac{1}{2} e^{ax} [\sin(bx + cx) + \sin(bx - cx)] = \frac{1}{2} [e^{ax} \sin(b + c)x + e^{ax} \sin(b - c)x]. \end{aligned}$$

$$\text{Now } D^n [e^{ax} \sin(bx + cx)] = (b^2 + c^2)^{n/2} e^{ax} \sin \left[bx + c + n \tan^{-1} \left(\frac{b}{a} \right) \right]$$

$$\therefore y^n = \frac{1}{2} \left[\begin{aligned} &\left\{ a^2 + (b + c)^2 \right\}^{n/2} e^{ax} \sin \left\{ (b + c)x + n \tan^{-1} \frac{(b + c)}{a} \right\} \\ &+ \left\{ a^2 + (b - c)^2 \right\}^{n/2} e^{ax} \sin \left\{ (b - c)x + n \tan^{-1} \frac{(b - c)}{a} \right\} \end{aligned} \right]$$

Example. If $y = e^{ax} \sin bx$, proved that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$.

Solution.

$$\text{Let } y = e^{ax} \sin bx, \quad \therefore y_1 = ae^{ax} \sin bx + be^{ax} \cos bx$$

$$\text{and } y_2 = a^2 e^{ax} \sin bx + 2abe^{ax} \cos bx - b^2 e^{ax} \sin bx \quad \dots\dots\dots(1)$$

$$\text{Also } -2ay_1 = -2a^2 e^{ax} \sin bx - 2abe^{ax} \cos bx \quad \dots\dots\dots(2)$$

$$\text{and } (a^2 + b^2)y = a^2 e^{ax} \sin bx + b^2 e^{ax} \sin bx \quad \dots\dots\dots(3)$$

Adding (1),(2)and (3), we get

$$y_2 - 2ay_1 + (a^2 + b^2)y = 0$$

LEIBNITZ'S THEOREM:

The find n^{th} differential coefficient of two function of x

If u and v are any two functions of x such that all their desired differential coefficients exist, then the n^{th} differential coefficient of their product is given by

$$D^n (uv) = (D^n u).v + {}^n C_1 D^{n-1} u.Dv + {}^n C_2 D^{n-1} u.D^2 v + \dots\dots\dots + {}^n C_r D^{n-r} u.D^r v + \dots\dots\dots + uD^n v.$$

OR

$$D^n (uv) = (D^n u).v + nD^{n-1} u.Dv + \frac{n(n-1)}{2!} D^{n-2} u.D^2 v + \dots\dots\dots + nDuD^{n-1} v + uD^n v.$$

Proof.

Let $y = uv$, we have

$$Dy(uv) = (D^n u) \cdot v + u \cdot Dv \quad \dots\dots\dots(1)$$

From (1) we see that the theorem is true for $n = 1$.

Now assume that the theorem is true for a particular value of n , we have

$$D^n(uv) = (D^n u) \cdot v + n_{c_1} D^{n-1} u \cdot Dv + n_{c_2} D^{n-2} u \cdot D^2 v + \dots\dots + n_{c_r} D^{n-r} u \cdot D^r v + n_{c_{r+1}} D^{n-r-1} u \cdot D^{r+1} v + \dots\dots + u D^n v \quad \dots\dots\dots(2)$$

Differentiating both side of (2) w,r,t,x, we get

$$D^{n+1}(uv) = \left[(D^{n+1} u) \cdot v + D^n u \cdot Dv \right] + \left(n_{c_1} D^n u \cdot Dv + n_{c_1} D^{n-1} u \cdot D^2 v \right) + \left(n_{c_2} D^{n-1} u \cdot D^2 v + n_{c_2} D^{n-2} u \cdot D^3 v \right) + \dots\dots + \left(n_{c_r} D^{n-r+1} u \cdot D^r v + n_{c_r} D^{n-r} u \cdot D^{r+1} v \right) + \left(n_{c_{r+1}} D^{n-r} u \cdot D^{r+1} v + n_{c_{r+1}} D^{n-r-1} u \cdot D^{r+2} v \right) + \dots\dots + \left(Du D^n v + u \cdot D^{r+1} v \right).$$

Rearranging the term, we get

$$D^{n+1}(uv) = (D^{n+1} u) \cdot v + (1 + n_{c_1}) (D^n u \cdot Dv) + (n_{c_1} + n_{c_2}) D^{n-1} u \cdot D^2 v + \dots\dots + (n_{c_r} + n_{c_{r+1}}) (D^{n-r} u \cdot D^{r+1} v) + \dots\dots + u D^{n+1} v \quad \dots(3)$$

But we know that $n_{c_1} + n_{c_{r+1}} = n_{c_{r+1}}$. Therefore

$1 + n_{c_1} = n_{c_0} + n_{c_1} = n_{c_1}$, $n_{c_1} + n_{c_2} = n_{c_2}$ and so on.

Hence(3) gives

$$D^{n+1}(uv) = (D^{n+1} u) \cdot v + n_{c_1} (D^n u) \cdot Dv + n_{c_2} (D^{n-1} u) \cdot (D^2 v) + \dots\dots\dots + n_{c_{r+1}} D^{n-r} u \cdot D^{r+1} v + \dots\dots + u \cdot D^{n+1} v. \quad \dots\dots\dots(4)$$

From (4) we see that if the theorem is true for any value of n , it is also true for the next value of n . But we have already seen that the theorem is true for $n = 1$. Hence it must be true for $n = 2$ and so for $n = 3$, and so on. Thus the Leibnitz's theorem is true for all positive integral values of n .

Example. Find the n th differential coefficients of

- (i) $\sin ax \cos bx$,
- (ii) $\log[(ax + b)(cx + d)]$.

Solution.

(i) Let $y = \sin ax \cos bx = \frac{1}{2}[2 \sin ax \cos bx] = \frac{1}{2}[2 \sin(a+b)x + \sin(a-b)x]$.

we know that $D^n \sin(ax+b) = a^n \sin\left(ax+b+\frac{1}{2}n\pi\right)$.

$$\therefore y^n = \frac{1}{2} \left[(a+b)^n \sin\left\{(a+b)x + \frac{1}{2}n\pi\right\} + (a-b)^n \sin\left\{(a-b)x + \frac{1}{2}n\pi\right\} \right].$$

(ii) Let $y = \log[(ax+b)(cx+d)] = \log(ax+b) + \log(cx+d)$.

We know that $D^n \log(ax+b) = (-1)^{n-1} (n-1)! a^n (ax+b)^{-n}$

$$\begin{aligned} \therefore y_n &= (-1)^{n-1} (n-1)! a^n (ax+b)^{-n} + (-1)^{n-1} (n-1)! c^n (cx+d)^{-n} \\ &= (-1)^{n-1} (n-1)! \left[\frac{a^n}{(ax+b)^n} + \frac{c^n}{(cx+d)^n} \right]. \end{aligned}$$

Example. Find the nth derivatives of $\frac{1}{1-5x+6x^2}$.

Solution.

$$\text{Let } y = \frac{1}{6x^2 - 5x + 1} = \frac{1}{(2x-1)(3x-1)}.$$

$$\therefore \frac{1}{6x^2 - 5x + 1} \equiv \frac{A}{2x-1} + \frac{B}{3x-1} \equiv \frac{A(3x-1) + B(2x-1)}{(2x-1)(3x-1)},$$

Putting $x = \frac{1}{2}, 1 = -\frac{B}{3}$, i.e. $B = -3$; putting $x = \frac{1}{3}, A = 2$.

$$\text{Hence } y = \frac{2}{2x-1} + \frac{3}{3x-1} = 2(2x-1)^{-1} - 3(3x-1)^{-1}$$

$$\text{Therefore } y_n = \frac{d^n}{dx^n} [2(2x-1)^{-1}] - \frac{d^n}{dx^n} [3(3x-1)^{-1}]$$

Now we apply the formula,

$$D^n (ax+b)^{-1} = (-1)^n (n!)(ax+b)^{-n-1} a^n.$$

$$\text{Hence } y_n = 2 \cdot 2^n (-1)^n (n!)(2x-1)^{-n-1} - 3 \cdot 3^n (-1)^n (n!)(3x-1)^{-n-1}.$$

$$\text{or } y_n = (-1)^n (n!) \left[\frac{2^{n+1}}{(2x-1)^{n+1}} + \frac{3^{n+1}}{(3x-1)^{n+1}} \right].$$

Example. If $y = \sin ax + \cos ax$, prove that $y^n = a^n [1 + (-1)^n \sin 2ax]^{1/2}$.

Let $y = \sin ax + \cos ax$, then

$$\begin{aligned} \therefore y_n &= a^n \sin\left(ax + \frac{1}{2}n\pi\right) + a^n \cos\left(ax + \frac{1}{2}n\pi\right) \\ &= a^n \left[\left\{ \sin\left(ax + \frac{1}{2}n\pi\right) + \cos\left(ax + \frac{1}{2}n\pi\right) \right\}^2 \right]^{1/2} \\ &= a^n \left[1 + 2 \sin\left(ax + \frac{1}{2}n\pi\right) \cos\left(ax + \frac{1}{2}n\pi\right) \right]^{1/2} \\ &= a^n [1 + \sin(2ax + n\pi)]^{1/2} = [1 + \sin n\pi \cos 2ax + \cos n\pi \sin 2ax]^{1/2} \\ y_n &= a^n [1 + (-1)^n \sin 2ax]^{1/2} \quad [\text{Q } \sin n\pi = 0 \text{ and } \cos n\pi = (-1)^n] \end{aligned}$$

Example. If $p^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$, prove that $p + \left(\frac{d^2 p}{d\theta^2}\right) = \frac{a^2 b^2}{p^3}$.

Solution.

$$\text{Given } p^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta \quad \dots(1)$$

Differentiating both sides of (1) w.r.t θ , we get

$$\therefore 2p \left(\frac{dp}{d\theta}\right) = 2(b^2 - a^2) \cos \theta \sin \theta \quad \dots(2)$$

Again differentiating both sides of (2) w.r.t θ , we get

$$p \left(\frac{d^2 p}{d\theta^2}\right) + \left(\frac{dp}{d\theta}\right)^2 = (b^2 - a^2)(\cos^2 \theta - \sin^2 \theta) \quad \dots(3)$$

Multiplying (3) by p^2 and substituting the value of $\frac{dp}{d\theta}$ from (1) and (3), we get

$$p^3 \left(\frac{d^2 p}{d\theta^2}\right) + (b^2 - a^2)^2 \cos^2 \theta \sin^2 \theta = p^2 (b^2 - a^2)(\cos^2 \theta - \sin^2 \theta)$$

$$\text{or } p^3 \left(\frac{d^2 p}{d\theta^2}\right) = (a^2 \cos^2 \theta + b^2 \sin^2 \theta)(b^2 - a^2)(\cos^2 \theta - \sin^2 \theta) - (b^2 - a^2)^2 \cos^2 \theta \sin^2 \theta$$

$$\begin{aligned}
\text{or } p^4 + p^3 \left(\frac{d^2 p}{d\theta^2} \right) &= (b^2 - a^2)[(\cos^2 \theta - \sin^2 \theta)(a^2 \cos^2 \theta + b^2 \sin^2 \theta) - (b^2 - a^2) \cos^2 \theta \sin^2 \theta] \\
&\quad + (a^2 \cos^2 \theta - b^2 \sin^2 \theta)^2 \\
&= (b^2 - a^2)[(a^2 \cos^4 \theta - b^2 \sin^4 \theta) + (a^2 \cos^2 \theta + b^2 \sin^2 \theta)^2] \\
&= b^2 a^2 (\cos^2 \theta + \sin^2 \theta) = a^2 b^2
\end{aligned}$$

Hence $p + \left(\frac{d^2 p}{d\theta^2} \right) = \frac{a^2 b^2}{p^3}$.

Example. Find y_n if $y = x^{n-1} \log x$.

Solution.

By Leibnitz's theorem, we get

$$\begin{aligned}
y_n = D^n (x^{n-1} \log x) &= D^n (x^{n-1}) \log x + n D^{n-1} (x^{n-1}) D \log x + \frac{n(n-1)}{2!} D^{n-2} (x^{n-1}) D^2 \log x \\
&\quad + \frac{n(n-1)(n-2)}{3!} D^{n-3} (x^{n-1}) D^3 \log x + \dots + x^{n-1} D^n \log x.
\end{aligned}$$

Now $D^n x^m = \frac{m!}{(m-n)!} x^{m-n}; D^n x^{n-1} = 0$

$$D^{n-1} x^m = \frac{m!}{(m-n+1)!} x^{m-n+1}; \quad \therefore D^{n-1} x^{n-1} = (n-1)!$$

$$D^{n-2} x^{n-1} = \frac{(n-1)!}{1!} x; D^{n-3} x^{n-1} = \frac{(n-1)!}{2!} x^2$$

and $D^n \log x = (-1)^{n-1} \frac{(n-1)!}{x^n}$

Hence

$$\begin{aligned}
y_n &= \left[n(n-1)! \frac{1}{x} + \frac{n(n-1)(n-1)!}{2! \cdot 1!} x \left(-\frac{!}{x^2} \right) + \frac{n(n-1)(n-2)(n-1)!}{3! \cdot 2!} x^2 \frac{2}{x^3} + \dots \right. \\
&\quad \left. \dots + x^{n-1} \frac{(-1)^{n-1} (n-1)!}{x^n} \right] \\
&= \frac{(n-1)!}{x} [1 - \{1 - {}^n c_1 - {}^n c_2 - {}^n c_3 + \dots + (-1)^{n+1} c_n\}] \\
&= \frac{(n-1)!}{x} [1 - (1-1)^n] = \frac{(n-1)!}{x}
\end{aligned}$$

Aliter. $y = x^{n-1} \log x \quad \therefore y_1 = (n-1)x^{n-2} \log x + x^{n-2}$.

$\therefore xy_1 = (n-1)x^{n-1} \log x + x^{n-1} = (n-1)y + x^{n-1}$.

Differentiating both sides $(n-1)$ times, we have

$$D^{n-2}(xy_1) = (n-1)D^{n-1}y + D^{n-1}x^{n-1}.$$

$\therefore xy_n + (n-1)y_{n-1} = (n-1)y_{n+1} + (n-1)! \quad \text{or} \quad y_n = \frac{(n-1)!}{x}$

Example.

If $y = a \cos(\log x) + b \sin(\log x)$, show that $x^2 y_2 + xy_1 + y = 0$

and $x^2 y_{n+2} + (2n-1)xy_{n+1} + (n^2 + 1)y_n = 0$.

Solution.

Let $y = a \cos(\log x) + b \sin(\log x)$,

$$y_1 = -a \sin(\log x) \cdot \frac{1}{x} + b \cos(\log x) \cdot \frac{1}{x} \quad \text{or} \quad xy_1 = -a \sin(\log x) + b \cos(\log x)$$

Now again differentiating both sides, we get

$$xy_2 + y_1 = -a \cos(\log x) \cdot \frac{1}{x} - b \sin(\log x) \cdot \frac{1}{x}$$

or $x^2 y_2 + xy_1 = -[a \cos(\log x) + b \sin(\log x)]$

or $x^2 y_2 + xy_1 = -y$

or $x^2 y_2 + xy_1 + y = 0$.

Again differentiating both sides in times by Leibnitz's theorem,

$$D^n(x^2 y_2) + D^n(xy_1) + D^n(y) = 0.$$

or $x^2 D^n y_2 + nDx^2 D^{n-1} y_2 + \frac{n(n-1)}{2} D^2 x^2 D^{n-2} y_2 + xD^n y_1 + nD^{n+1} y_1 + y_n = 0$

or $x^2 y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n + y_n = 0$

or $x^2 y_{n+2} + (2n-1)xy_{n+1} + (n^2 + 1)y_n = 0$.

Example If $y = \sin(m \sin^{-1} x)$. prove that $(1-x^2)y_2 - xy_1 + m^2 y = 0$ and deduce that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - m^2)y_n = 0$.

Solution: Let $y = \sin(m \sin^{-1} x)$.

$$y_1 = \cos(m \sin^{-1} x) \cdot \frac{m}{\sqrt{(1-x^2)}} \quad \text{or} \quad (1-x^2)y_1^2 = m^2 \cos^2(m \sin^{-1} x).$$

$$\text{or } (1-x^2)y_1^2 = m^2 - m^2 \sin^2(m \sin^{-1} x) = m^2 - m^2 y^2$$

$$\therefore (1-x^2)y_1^2 + m^2 y = m^2.$$

Again differentiating both sides, we have

$$2y_1 y_2 (1-x^2) - 2xy_1^2 + 2m^2 y y_1 = 0. \text{ or } y_2 (1-x^2) x y_1 + m^2 y = 0.$$

Now differentiating n time by Leibnitz's theorem, we get

$$y_{n+2} (1-x^2) + n y_{n+1} (-2x) + \frac{n(n-1)}{2!} y_n (-2) - x y_{n+1} - n y_n + m^2 y_n = 0,$$

$$\text{or } (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - m^2)y_n = 0.$$

To find The nth Derivative When x = 0

Example: Find $(y_n)_0$. if $y = \sin(a \sin^{-1} x)$.

Solution:

Let $y = \sin(a \sin^{-1} x)$(1)

$$\therefore y_1 = \cos(a \sin^{-1} x) \cdot \frac{a}{\sqrt{(1-x^2)}},$$

$$\text{or } y_1^2 (1-x^2) = a^2 \cos^2(a \sin^{-1} x) = a^2 - a^2 \sin^2(a \sin^{-1} x) = a^2 - a^2 y^2$$

$$\text{or } y_1^2 (1-x^2) + a^2 y^2 - a^2 = 0. \quad \text{.....(2)}$$

Differentiating (2), we have

$$2y_1 y_2 (1-x^2) - y_1^2 (-2x) + 2a^2 y y_1 = 0.$$

$$\text{or } y_2 (1-x^2) + x y_1 + a^2 y_1 = 0 \quad \text{.....(3)}$$

Differentiating (3) n times, we have

$$y_{n+2} (1-x^2) - n y_{n+1} 2x - \frac{n(n-1)}{2!} y_n \cdot 2 - x y_{n+1} - n y_n + a^2 y_n = 0.$$

$$\text{or } y_{n+2} (1-x^2) - (2n+1) x y_{n+1} - (n^2 - a^2) y_n = 0. \quad \text{.....(4)}$$

Putting $x = 0$ in (1), we get $(y)_0 = 0$.

Putting $x = 0$ in (2), we get $(y_1)_0 = 0$

Putting $x = 0$ in (3), we get $(y_2)_0 = 0$ and

Putting $x = 0$ in (4), we get $(y_{n+2})_0 = (n^2 - a^2)(y_n)_0$

Now putting $n = 2$ in (5), $(y_6)_0 = (2^2 - a^2)(y_2)_0 = 0$.

Putting $n = 4$ in (5), $(y_6)_0 = (4^2 - a^2)(y_4)_0 = 0$.

Similarly $(y_8)_0 = 0$.

Thus the derivatives for which n is even are zero

Again, putting $n = 1$, $(y_3)_0 = (1^2 - a^2)(y_1)_0 = (1^2 - a^2)a$.

Now when n is odd. $(y_{n+2})_0 = (n^2 - a^2)(y_n)_0$.

Putting n is place of $(n-2)$ we obtain

$$\begin{aligned} (y_n)_0 &= [(n-2)^2 - a^2](y_{n-2})_0 \\ &= [(n-2)^2 - a^2][(n-4)^2 - a^2][(n-6)^2 - a^2] \dots [3^2 - a^2][y_3]_0 \\ &= [(n-2)^2 - a^2][(n-4)^2 - a^2] \dots [3^2 - a^2][1^2 - a^2].a \end{aligned}$$

Example.

If $y = \tan^{-1} x$, prove that $(1 + x^2)y_2 + 2xy_1 = 0$ and deduce that

$$(1 + x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0 \text{ Hence determine } (y_n)_0$$

Solution.

Let $y = \tan^{-1} x$ (1)

$$\therefore y_1 = \frac{1}{(1+x^2)}, \dots(2)$$

or $(1+x^2)y_1 - 1 = 0$(3)

Differentiating (3), we get $(1+x^2)y_2 + 2xy_1 = 0$ (4)

Now , differentiating (4) n times by Leibnitz's theorem, we get

$$y_{n+2}(1+x^2) + ny_{n+1}(2x) + \frac{n(n-1)}{2!}y_n \cdot 2 + 2xy_{n+1} + 2ny_n = 0$$

or $(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$ (5)

Putting $x = 0$, in (1),(2) and (4), we get

$$(y)_0 = 0, (y_1)_0 = 1, (y_2)_0 = 0.$$

Also putting $x = 0$ in (5) we get

$$(y_{n+2})_0 = -[(n+1)n](y_n)_0. \dots\dots\dots(6)$$

Putting $n-2$ in place of n in the formula (6), we get

$$\begin{aligned} (y_n)_0 &= [(n-1)(n-2)](y_{n-2})_0 \\ &= [-\{(n-1)(n-2)\}][-\{(n-3)(n-4)\}](y_{n-4})_0 \end{aligned}$$

Since from (6), we have $(y_{n-2})_0 = -\{(n-3)(n-4)\}(y_{n-4})_0$

Case I. When n is even, we have

$$\begin{aligned} (y_n)_0 &= [-\{[(n-1)(n-2)]\}][-\{(n-3)(n-4)\}] \dots [-\{(3)(2)\}](y_2)_0 \\ &= 0, \quad \text{Since } (y_2)_0 = 0. \end{aligned}$$

Case II. When n is odd, we have

$$\begin{aligned} (y_n)_0 &= [-\{[(n-1)(n-2)]\}][-\{(n-3)(n-4)\}] \dots [-\{(4)(3)\}][-\{(2)(1)\}](y_1)_0 \\ &= (-1)^{(n-1)^2} (n-1)!, \quad \text{since } (y_1)_0 = 1. \end{aligned}$$

ADDITIONAL PROBLEMS:

1. Find the n th differential coefficient of

(i) $\sin^3 x$

(ii) $\sin x \cos 3x$

(iii) $e^{ax} \cos^2 x \sin x$

(iv) $\frac{x^2}{(x+2)(2x+3)}$

2. If $y = e^{ax} \sin bx$, prove that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$

3. If $y = \cos(m \sin^{-1} x)$, prove that $(1 - x^2)y_2 - xy_1 + m^2y = 0$ and
 $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} + (m^2 - n^2)y_n = 0$

4. If $y = (\sin^{-1} x)^2$, prove that $(1 - x^2)y_2 - xy_1 - 2 = 0$ and deduce that
 $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2y_n = 0$

5. If $y = e^{\tan^{-1} x}$, prove that $(1 + x^2)y_{n+2} + \{2(n + 1)x - 1\}y_{n+1} + n(n + 1)y_n = 0$